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E: ISSN NO.: 2349-980X **Shrinkhla Ek Shodhparak Vaicharik Patrika**

The Hopf Bifurcation Dynamics of Food Web of Two Logistic Prey and Modified Leslie- Gower Type Predator

Abstract

The mathematical model in the form of food web consisting of two logistic preys and a predator is taken from [1]. Modified Leslie-Gower type dynamics is considered for the predator [1]. The model is analyzed mathematically [1]. Analysis of nonzero positive equilibrium gives conditions for persistence [1]. The existence of Hopf bifurcation has been observed in range of biological feasible values for the key parameters. This paper is the extended part of published research paper [1].

Keywords: Range of Biological Feasible Parameters, Hopf Bifurcation Dynamics.

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Introduction

It is known that by the study of various research papers [1-22] that the nature is nonlinear and is observed multi rich dynamics. In the research paper [1] for the given biological food system global and local behaviour has been investigated under the biological feasible range of parameters. In this paper it is observed existence of hopf bifurcation analytical and numerical.

Review of Literature

A lot of research work has been carried out on ecological systems comprising of food chains and food web of variable lengths [01-22]. The underlying nonlinear equations have complex dynamical behavior: hopf bifurcation, quasi-periodic behavior. The chaos is not frequently observed and the models reveal quasi periodic nature of the solution. Due to indirect competition between two predator species, one or more species may undergo extinction.

Aim of the Study

We have to study quasiperiodic, hopf bifurcation solutions of the nonlinear model [1].

The Mathematical Model

The transformed non-dimensional form of the biological food web [1] is given

$$\frac{dy_1}{dt} = y_1\left(1 - y_1 - \frac{w_1 y_3}{1 + w_2 y_1 + w_3 y_2}\right) = y_1 f_1(y_1, y_2, y_3)$$

$$\frac{dy_2}{dt} = y_2\left[(1 - y_2)w_4 - \frac{w_5 y_3}{1 + w_3 y_2 + w_2 y_1}\right] = y_2 f_2(y_1, y_2, y_3) \qquad (1)$$

$$\frac{dy_3}{dt} = w_6 y_3^2\left(1 - \frac{w_7}{1 + w_8 y_1 + w_9 y_2}\right) = w_6 y_3^2\left(1 - \frac{w_7}{1 + \alpha_1 w_2 y_1 + \alpha_2 w_3 y_2}\right) = y_3 f_3(y_1, y_2, y_3)$$

$$w_i > 0, i = 1,2,3,4,5,6,7; \quad y_i \geq 0, \ i = 1,2,3; \ \alpha_1 \neq \alpha_2 .$$

Mathematical Analysis

The system can be splitted into two disconnected Kolmogorov food sub webs [1]

Lemma 1

Consider the domain $D_1 = \{(y_1, y_3): 0 < y_1 < \overline{y}_1 < 1, 0 < y_3\}$ and $D_2 = \{(y_2, y_3): 0 < y_2 < \overline{y}_2 < 1, 0 < y_3\}$, he sub webs of (1) in [1] is Kolmogorov in the domain D1 and domain D2 under the following conditions:

$$w_7 / (1+\alpha_1 w_2 \overline{y}_1)^2 < 1 < w_7 / (1+\alpha_1 w_2 \overline{y}_1) < w_7 \qquad (2)$$

$$w_7 / (1+\alpha_2 w_3 \overline{y}_2)^2 < 1 < w_7 / (1+\alpha_2 w_3 \overline{y}_2) < w_7 \text{ respectively} \qquad (3)$$

The proof of the theorem given for existence of positive equilibrium point and stability is established in [1]

Theorem

The system (1) has positive equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ under (2) and (3) provided one of the following is satisfied [1]:

$$w_3(w_1 w_4 - w_5) < \frac{w_1 w_4 \varepsilon}{\alpha_2}; \varepsilon = w_7 - 1 \qquad (4)$$

$$w_2(w_5 - w_1 w_4) < \frac{\varepsilon w_5}{\alpha_1} \qquad (5)$$

Theorem

The positive equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is locally asymptotically stable provided the following are satisfied simultaneously [1]:

$$2(w_1 w_4 \varepsilon + w_5 w_3 \alpha_2) + \frac{\Delta}{w_2} > \Delta + 2\alpha_2 w_3 w_1 w_4 \qquad (6)$$

$$w_3(w_5 \varepsilon + \alpha_1 w_1 w_4 w_2) + \Delta > \frac{w_3 w_5}{w_1 w_4}\Delta + \alpha_1 w_2 w_5 w_3 \qquad (7)$$

$$\varepsilon = w_7 - 1 > 0; \quad \Delta = \alpha_1 w_1 w_2 w_4 + \alpha_2 w_3 w_5$$

The following theorem [1] gives the conditions for the global stability of positive nonzero equilibrium point.

Theorem

The positive equilibrium point $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is globally asymptotically stable provided the following are satisfied[1]:

$$A = (1 + w_2\hat{y}_1 + w_3\hat{y}_2 - w_2) > 0; \; B = (1 + w_2\hat{y}_1 + w_3\hat{y}_2 - w_3) > 0. \qquad w_3^2 m^2 + w_2^2 w_4^2 < 4 m w_4 A B;$$

$$m = \frac{\alpha_1 w_2 w_5}{\alpha_2 w_1 w_3} \qquad (8)$$

Hopf's Analysis of the Food Web

Assume $y_1 = \hat{y}_1 + u, y_2 = \hat{y}_2 + v, y_3 = \hat{y}_3 + w$, where u, v and w small perturbations are. The variational matrix about $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -\hat{y}_1 + \dfrac{w_1 w_2 \hat{y}_1 \hat{y}_3}{(1 + w_2\hat{y}_1 + w_3\hat{y}_2)^2} & \dfrac{w_1 w_3 \hat{y}_1 \hat{y}_3}{(1 + w_2\hat{y}_1 + w_3\hat{y}_2)^2} & -\dfrac{w_1 \hat{y}_1}{(1 + w_2\hat{y}_1 + w_3\hat{y}_2)} \\ \dfrac{w_2 w_5 \hat{y}_2 \hat{y}_3}{(1 + w_2\hat{y}_1 + w_3\hat{y}_2)^2} & \hat{y}_2\left(-w_4 + \dfrac{w_5 w_3 \hat{y}_3}{(1 + w_2\hat{y}_1 + w_3\hat{y}_2)^2}\right) & -\dfrac{w_5 \hat{y}_2}{(1 + w_2\hat{y}_1 + w_3\hat{y}_2)} \\ \dfrac{\alpha_1 w_7 w_2 w_6 \hat{y}_3 \hat{y}_3}{(1 + \alpha_1 w_2\hat{y}_1 + \alpha_2 w_3\hat{y}_2)^2} & \dfrac{\alpha_2 w_7 w_3 w_6 \hat{y}_3 \hat{y}_3}{(1 + \alpha_1 w_2\hat{y}_1 + \alpha_2 w_3\hat{y}_2)^2} & 0 \end{bmatrix}$$

The characteristic equation of variational matrix is

$$\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2 = 0 \qquad (9)$$

where $a_0 = -(a_{11} + a_{22}); a_1 = (a_{11}a_{22} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31});$

$$a_2 = (a_{13}a_{31}a_{22} + a_{23}a_{32}a_{11} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}).$$

Let $a_{11} = -m_1; a_{22} = -m_2; \; a_{13} = -m_3; \; a_{23} = -m_4$, then

$a_0 = m_1 + m_2 > 0; a_1 = (m_1 m_2 - a_{12} a_{21} + m_4 a_{32} + m_3 a_{31}) > 0;$

$a_2 = (m_3 a_{31} m_2 + m_4 a_{32} m_1 + a_{12} m_4 a_{31} + a_3 a_{21} a_{32}) > 0.$

Applying Routh's criteria $a_0 > 0$ provided $(a_{11} + a_{22}) < 0$, that is, $a_{11} < 0$, $a_{22} < 0$. Also $a_2 > 0$, $a_1 > 0$ and $a_1 a_0 - a_2 > 0$. Therefore positive nonzero equilibrium point is locally asymptotically stable. None of the roots of equation (9) is zero as $a_0 \neq 0$. Substituting $\lambda = \pm i\omega$ into (19), the real and imaginary partitions of the results lead to the following conditions: (i) $\omega = \pm\sqrt{a_1}$ (ii) $a_0 \omega^2 = a_2$

(i) and (ii) and (19) results that a pair of purely imaginary roots $\pm i\sqrt{a_1}$ and a real root " $-a_0$ ".

Transversality condition: - Let the characteristic equationbesuch that it contains a real root, say c_1, and a pair of purely imaginary roots $\lambda_1 = \lambda_1' \pm i\lambda_2'$:

$(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)(\lambda - c_1) = 0.$

or $\lambda^3 - (2\lambda_1' + c_1)\lambda^2 + (|\lambda_1|^2 + 2\lambda_1' c_1) - |\lambda_1|^2 c_1 = 0$ (10)

Comparing the coefficients of (15) and (16) gives

$a_1(-a_0 - 2\lambda_1') = -a_2 + 2\lambda_1'(2\lambda_1' + a_0)^2$ (11)

Differentiating (11) with respect to bifurcation parameter w_7, and substituting $w_7 = w_7^*$ and $\lambda_1'(w_7^*) = 0$ yields [11]:

$$\left.\frac{\partial \lambda_1'}{\partial w_7}\right|_{w_7 = w_7^*} = -\frac{(a_0 \frac{\partial a_1}{\partial w_7} + a_1 \frac{\partial a_0}{\partial w_7} - \frac{\partial a_2}{\partial w_7})}{2(a_0^2 + a_1)} \quad (12)$$

$$\begin{aligned}\frac{\partial a_2}{\partial w_7} - a_0\frac{\partial a_1}{\partial w_7} - a_1\frac{\partial a_0}{\partial w_7} = \{&\frac{\partial m_1}{\partial w_7}(a_{12}a_{21} - 2m_1 m_2 - m_2^2 - m_3 a_{31}) \\ &+ \frac{\partial m_2}{\partial w_7}(a_{12}a_{21} - 2m_1 m_2 - m_1^2 - m_4 a_{32}) + \frac{\partial m_3}{\partial w_7}(a_{21}a_{32} - a_{31}m_1) \\ &+ \frac{\partial m_4}{\partial w_7}(a_{12}a_{31} - a_{32}m_2) + \frac{\partial a_{12}}{\partial w_7}(a_{31}m_4 + a_{21}a_0) + \frac{\partial a_{21}}{\partial w_7}(a_{32}m_3 + a_{12}a_0) \\ &+ \frac{\partial a_{31}}{\partial w_7}(a_{12}m_4 - m_1 m_3) + \frac{\partial a_{32}}{\partial w_7}(a_{21}m_3 - m_2 m_4)\};\end{aligned} \quad (13)$$

$$\frac{\partial \hat{y}_1}{\partial w_7} = \frac{w_1 w_4}{\Delta}; \frac{\partial \hat{y}_2}{\partial w_7} = \frac{w_5}{\Delta}; \frac{\partial \hat{y}_3}{\partial w_7} = \frac{\delta w_4}{\Delta^2}(w_1 w_4 w_2 + w_3 w_5). \quad (14)$$

$$\frac{\partial m_1}{\partial w_7} = \frac{\hat{y}_1 w_1 w_2}{l_1^3}\left[\left(\frac{w_1 w_2 w_4}{\Delta} + \frac{w_3 w_5}{\Delta}\right)\left(2\hat{y}_3 - \frac{l_1 \delta w_4}{\Delta}\right)\right] + \frac{w_1 c_1 w_4}{\Delta}, \quad (15)$$

$$\frac{\partial m_2}{\partial w_7} = \frac{\hat{y}_1 w_3 w_5}{l_1^3}\left[\left(\frac{w_1 w_2 w_4}{\Delta} + \frac{w_3 w_5}{\Delta}\right)\left(2\hat{y}_3 - \frac{l_1 \delta w_4}{\Delta}\right)\right] + \frac{w_1 c_2 w_4}{\Delta}, \quad (16)$$

$$\text{where } c_1 = \left(1 - \frac{\hat{y}_3 w_1 w_2}{l_1^2}\right) > 0,\ c_2 = \left(w_4 - \frac{\hat{y}_3 w_5 w_3}{l_1^2}\right) > 0$$

$$\frac{\partial m_3}{\partial w_7} = \frac{w_1}{\Delta l_1^2}[w_1 w_4 + w_3(w_1 w_4 - w_5)],$$
$$\frac{\partial m_4}{\partial w_7} = \frac{w_5}{\Delta l_1^2}[w_5 + w_2(w_5 - w_1 w_4)] \qquad (17)$$

$$\frac{\partial a_{12}}{\partial w_7} = \frac{w_3 w_1}{l_1^2}\left[\frac{w_1 w_4}{\Delta}\hat{y}_3 + \left(\frac{w_2 w_1 w_4 \hat{y}_1}{\Delta l_1} + \frac{w_3 w_5 \hat{y}_1}{l_1 \Delta}\right)\left(\frac{l_1 \delta w_4}{\Delta} - 2\hat{y}_3\right)\right],$$
$$\frac{\partial a_{21}}{\partial w_7} = \frac{w_2 w_5}{l_1^2}\left[\frac{w_5}{\Delta}\hat{y}_3 + \left(\frac{w_5 w_3 \hat{y}_2}{\Delta l_1} + \frac{w_1 w_2 w_4 \hat{y}_2}{l_1 \Delta}\right)\left(\frac{l_1 \delta w_4}{\Delta} - 2\hat{y}_3\right)\right] \qquad (18)$$

$$\frac{\partial a_{31}}{\partial w_7} = \frac{2\alpha_1 w_7 w_6 w_2 \hat{y}_3}{l_2^3 \Delta}\left[w_3 w_5\left(\frac{\delta w_4 l_2}{\Delta} - \alpha_2 \hat{y}_3\right) + w_1 w_2 w_4\left(\frac{l_2 \delta w_4}{\Delta} - \alpha_1 \hat{y}_3\right)\right],$$
$$\frac{\partial a_{32}}{\partial w_7} = \frac{2\alpha_2 w_7 w_6 w_3 \hat{y}_3}{l_2^3 \Delta}\left[w_3 w_5\left(\frac{\delta w_4 l_2}{\Delta} - \alpha_2 \hat{y}_3\right) + w_1 w_2 w_4\left(\frac{l_2 \delta w_4}{\Delta} - \alpha_1 \hat{y}_3\right)\right] \qquad (19)$$

It is observed that

$$\left.\frac{\partial \lambda_1'}{\partial w_7}\right|_{w_7 = w_7^*} \neq 0 .$$

Thus the transversality condition is satisfied. So a family of periodic solutions bifurcating from E_4 in the neighborhood of w_7^* exists, that is, the Hopf bifurcation will occur when $w_7 \in (w_7^* - \delta, w_7^* + \delta)$.

Numerical Simulation

Numerical simulations of the underlying non-linear system are carried out. The numerical values for various parameters are selected according to the mathematical restrictions obtained from the Kolmogorov analysis. These ensure that the parameters take biologically relevant values only.

Numerical results for hopf analysis with respect to w_7 is shown for the following data:

$$w_1 = 3.3, w_2 = 1.2, w_3 = 1.3, w_4 = 1.1, w_5 = 2.5,$$
$$w_6 = 1.0, w_7 = 1.6, \alpha_1 = 0.9, \alpha_2 = 0.3 \qquad (20)$$

The nonzero positive equilibrium point $(0.3549, 0.5557, 0.4200)$ is asymptotically stable as $a_1 a_0 - a_2 = 0.0671 > 0$. The analysis has established the existence of Hopf bifurcation. To get the value of w_7 where it occurs, the values of expression $a_1 a_0 - a_2$ are computed as function of w_7 and are shown in the table 1. The Hopf bifurcation occurs in the neighborhood of $w_7 = 1.35$. The variational matrix at the Hopf bifurcation point $w_7 = 1.35$ is given as

$$\begin{bmatrix} -0.0735 & 0.1040 & -0.3179 \\ 0.1836 & -0.2719 & -0.6081 \\ 0.1569 & 0.0567 & 0 \end{bmatrix}$$

Table 1: Values of $a_1 a_0 - a_2$ vs w_7

w_7	1.40	1.38	1.37	1.36	1.35	1.34	1.29
$a_1 a_0 - a_2$	0.0054	0.0029	0.0019	9.36e-04	1.14e-04	-5.939e-04	-0.0026

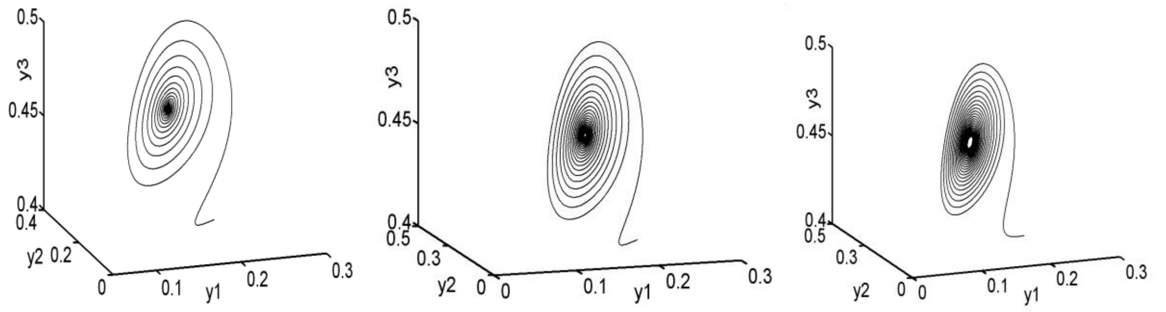
The eigenvalues of the above variational matrix are $-0.0003 + 0.2916i$, -0.0003 -0 .2916i, and -0.3448. The eigenvectors corresponding to eigenvectors at the hopf bifurcation point w_7 =1.35 are given as

0.5169 - 0.2759i	0.5169 + 0.2759i	0.3517
0.6826	0.6826	-0.9361
-0.1488 - 0.4107i	-0.1488 + 0.4107i	-0.0061

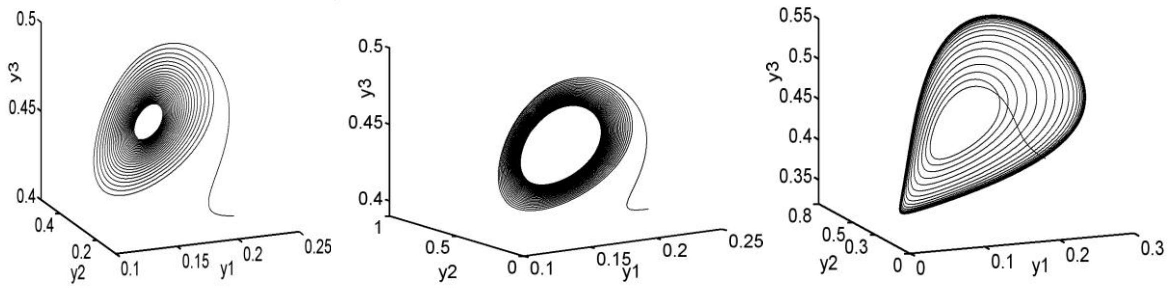
Transversality condition at the hopf bifurcation point w_7 =1.35 is computed as

$$\left.\frac{\partial \lambda_1'}{\partial w_7}\right|_{w_7 = w_7^*} = -\frac{\left(a_0 \frac{\partial a_1}{\partial w_7} + a_1 \frac{\partial a_0}{\partial w_7} - \frac{\partial a_2}{\partial w_7}\right)}{2(a_0^2 + a_1)} = -15.6933 \neq 0.0.$$

The transition in the global behavior in phase space is shown in fig 1. The fig. 2 show the three time series for w_7 = 1.29. The equilibrium point is stable at w_7 = 1.40 [1], while there is a limit cycle for w_7 = 1.29.



(a) w_7 =1.38 (b) w_7 =1.37 (c) w_7 =1.36



(d) w_7 =1.35 (e) w_7 =1.34 (f) w_7 =1.29

Fig. 1. Different phase plots with varying values of w_7

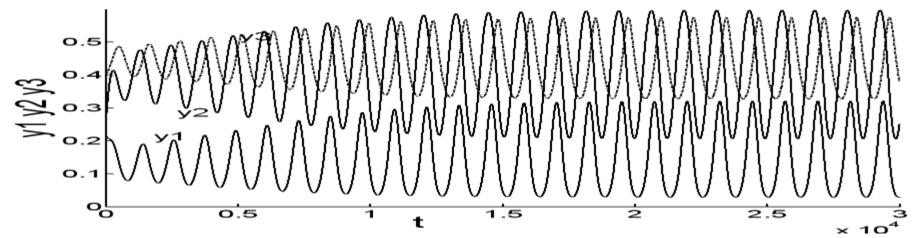


Fig. 2. Time series for w_7 =1.29

Conclusion

Further, it is observed that quasi-periodic behavior is obtained instead of limit cycle due to relaxation of the constraint considered.The existence of Hopf bifurcation analysis with varying key parameter is investigated numerically and analytically. Numerical integration of the food-web non-linear system is carried out under the Kolmogorov biologically feasible conditions. The limit cycle attractor is found.

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